Model-Order Estimation via Spectral-Modes

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Background & Motivation

Motivation

Exponential sums

Consider the classical model for noisy observations:

$$y(n) = \sum_{i=1}^{M} b_i z_i^n + w(n),$$

where $z_i = e^{(-\alpha_i + j\theta_i)}$, $w \sim N(0, \sigma_w^2)$

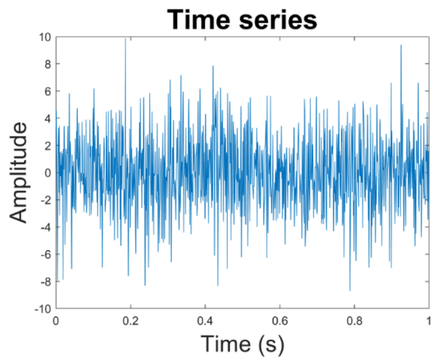
Goal: Accurately estimate all **signal parameters**.

This is a fundamental detection-estimation problem.

Existing approaches

Classical

Algebraic (super-resolution)

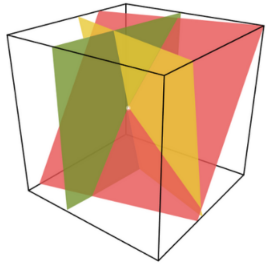


PARAMETER ESTIMATION FROM EIGENVALUES

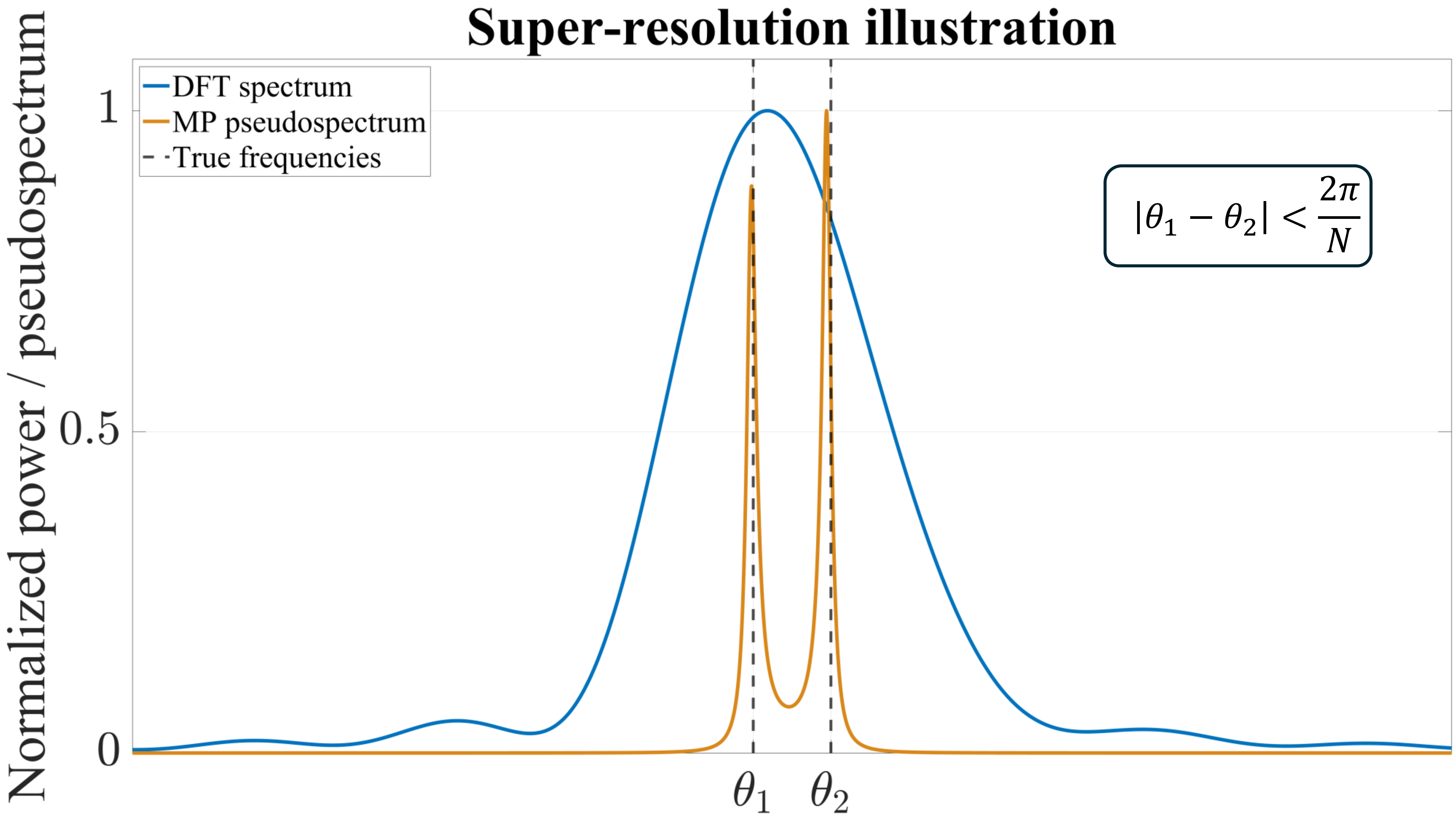
Hankel Matrix

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & & & \vdots \\ a_2 & & & & a_{2n-4} \\ \vdots & & & a_{2n-4} & a_{2n-3} \\ a_{n-1} & \cdots & a_{2n-4} & a_{2n-3} & a_{2n-2} \end{bmatrix}.$$

EVD



Existing approaches



The Matrix-Pencil (MP) Methode

The noiseless case

$$y(n) = \sum_{i=1}^{M} b_i z_i^n \qquad (1)$$

$$\boldsymbol{Y}_0, \boldsymbol{Y}_1 \in \mathbb{C}^{(N-L)\times L}$$

$$rank(\boldsymbol{Y}_0) = rank(\boldsymbol{Y}_1) = M$$

$$\boldsymbol{Y}_0 = \boldsymbol{Z}_L \boldsymbol{B} \boldsymbol{Z}_R, \qquad \boldsymbol{Y}_1 = \boldsymbol{Z}_L \boldsymbol{B} \boldsymbol{Z} \boldsymbol{Z}_R$$

$$\boldsymbol{Y}_1 - \lambda \boldsymbol{Y}_0 = \boldsymbol{Z}_L \boldsymbol{B} (\boldsymbol{Z} - \lambda \boldsymbol{I}) \boldsymbol{Z}_R \qquad (2)$$

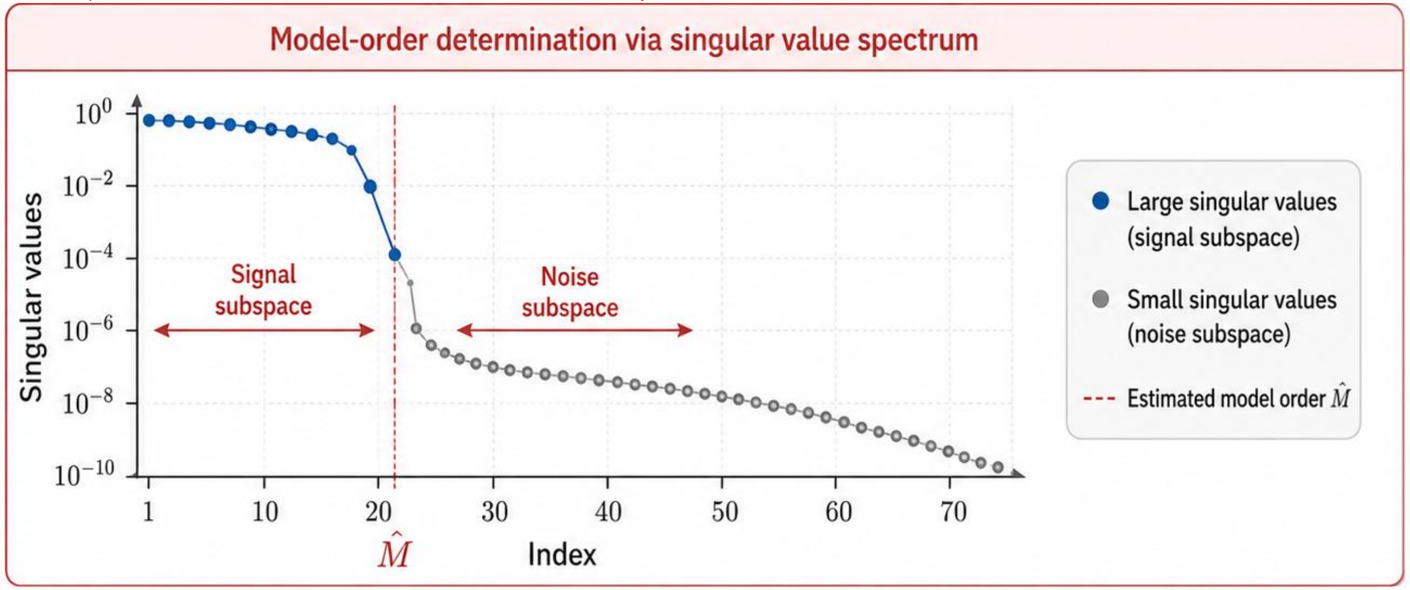
$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-L-1} & z_2^{N-L-1} & \cdots & z_M^{N-L-1} \end{bmatrix}}_{(N-L)\times M}^{Z_L}
\quad
\underbrace{\begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_M \end{bmatrix}}_{\substack{M \times M \\ \text{(diagonal)}}}^{B}
\quad
\underbrace{\begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_M \end{bmatrix}}_{\substack{M \times M \\ \text{(diagonal)}}}^{Z}
\quad
\underbrace{\begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{L-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{L-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_M & z_M^2 & \cdots & z_M^{L-1} \end{bmatrix}}_{M \times L}^{Z_R}$$

The Matrix-Pencil (MP) Methode

In the presence of noise

$$rank(\boldsymbol{Y}_0) = rank(\boldsymbol{Y}_1) = \mathrm{L}$$

- $L - M$ extraneous, **noise related components.**
- $\boldsymbol{Y}_0 \approx \boldsymbol{U}_{\widehat{M}} \boldsymbol{\Sigma}_{\widehat{M}} \boldsymbol{V}_{\widehat{M}}^H$



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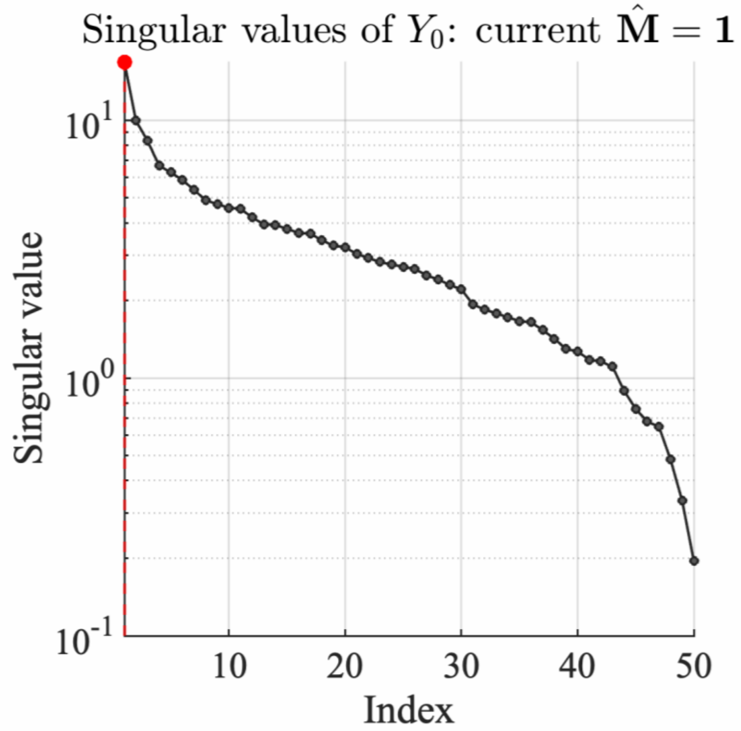
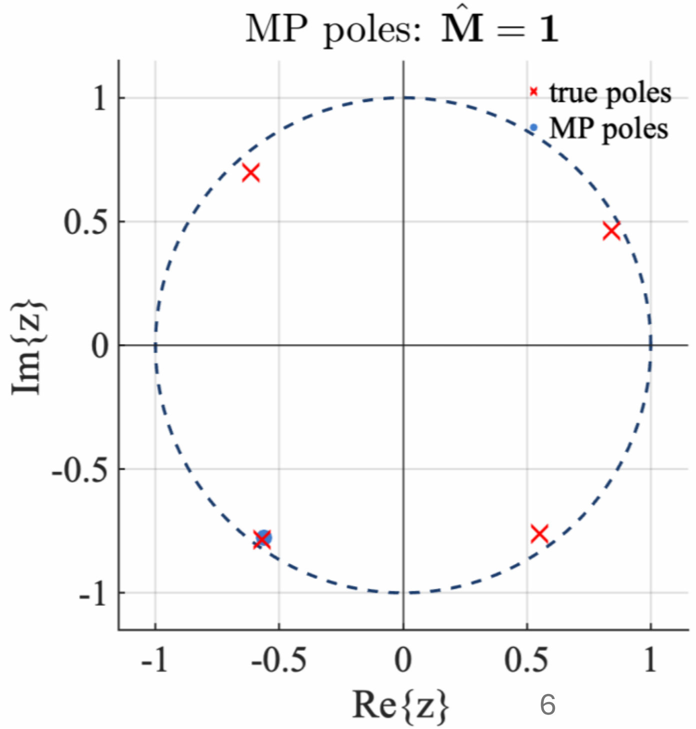




Extending the MP theorem: Accounting for noise

In presence of noise

$$y(n) = \sum_{i=1}^{M} b_i z_i^n + w(n)$$

$$\boldsymbol{Y}_0, \boldsymbol{Y}_1 \in \mathbb{C}^{(N-L)\times L}$$

$$rank(\boldsymbol{Y}_0) = rank(\boldsymbol{Y}_1) = L$$

Theorem 1.

$$\mathbf{Y}_0 = \boldsymbol{\Phi}_L \boldsymbol{\mathcal{B}} \boldsymbol{\Phi}_R, \qquad \mathbf{Y}_1 = \boldsymbol{\Phi}_L \boldsymbol{\mathcal{B}} \boldsymbol{\Lambda} \boldsymbol{\Phi}_R$$

$\mathcal{B}, \Lambda \in \mathbb{C}^{L\times L}$ are diagonal matrices containing the estimated amplitudes and poles.

$$\boldsymbol{\Phi}_L = \begin{bmatrix} \underbrace{\boldsymbol{Z}_L + \boldsymbol{E}_L}_{M \text{ columns}} , & \boldsymbol{C}_L \end{bmatrix}, \quad \boldsymbol{\mathcal{B}} = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_L \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L \end{bmatrix}, \quad \boldsymbol{\Phi}_R = \begin{array}{l} \left[\begin{array}{c} \boldsymbol{Z}_R + \boldsymbol{E}_R \\ \boldsymbol{C}_R \end{array}\right] \begin{array}{l} \}\, M \text{ rows} \\ \end{array} \end{array}$$

$(N-L)\times L$ | $L \times L$ (diagonal) | $L \times L$ (diagonal) | $L \times L$

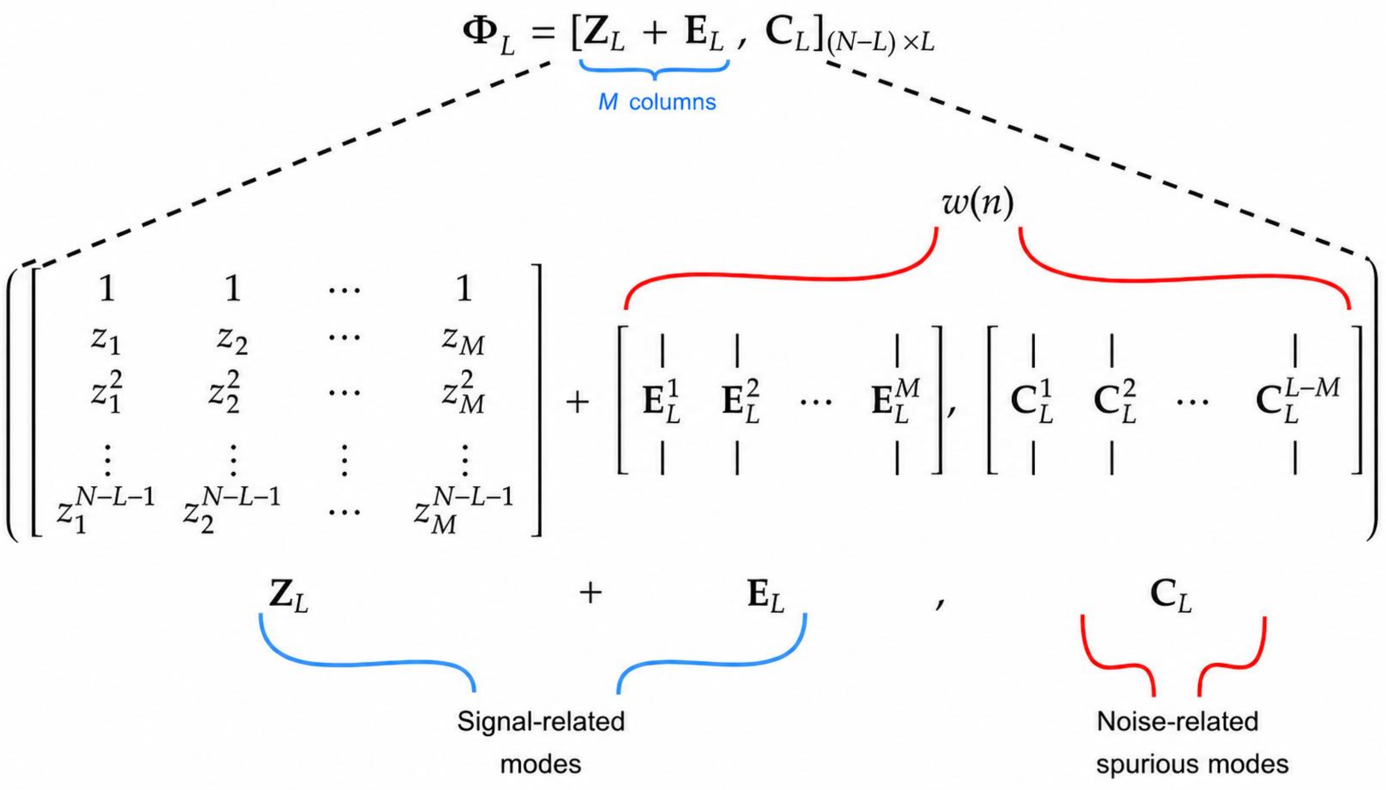
- Y. -I. Segman, A. Amar and R. Talmon, "Structure-Aware Matrix Pencil for Estimating the Number of Damped Complex Exponentials," in *IEEE Transactions on Signal Processing*, vol. 74, pp. 2461-2475, 2026

The Spectral-Modes

- $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$ are termed left and right **spectral-modes**.

- $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$ are also the **pseudo inverse** matrices of the left and right **eigenvector matrices**.

- The L left modes can be divided into two types: $\boldsymbol{M}$ **signal modes**, and $\boldsymbol{L-M}$ **noise modes**.



- It was shown in a subsequent work that $\boldsymbol{C}_L$ also has an obscured Vandermonde structure, but less pronounced.

Introducing Spectral-Modes

Proposition 1: Structure of a signal-mode error

A signal-related mode is approximately a Vandermonde vector corrupted by leakage and bounded noise.

$$E_L^i = \underbrace{\sum_{m\neq i} \gamma_{i,m}\, a(z_m)}_{\text{leakage from other poles}} + \xi_i$$

$$a(z) = [1, z, z^2, \dots, z^{N-L-1}]^T$$

Bounds *with probability at least $1-\epsilon$*

$$|\gamma_{i,m}| \leq 2\Delta_i \sqrt{\frac{2\log(\epsilon^{-1})}{SNR_i}}, \qquad \|\xi_i\|_\infty \leq \sqrt{\frac{2\log(\epsilon^{-1})}{SNR_i}}$$

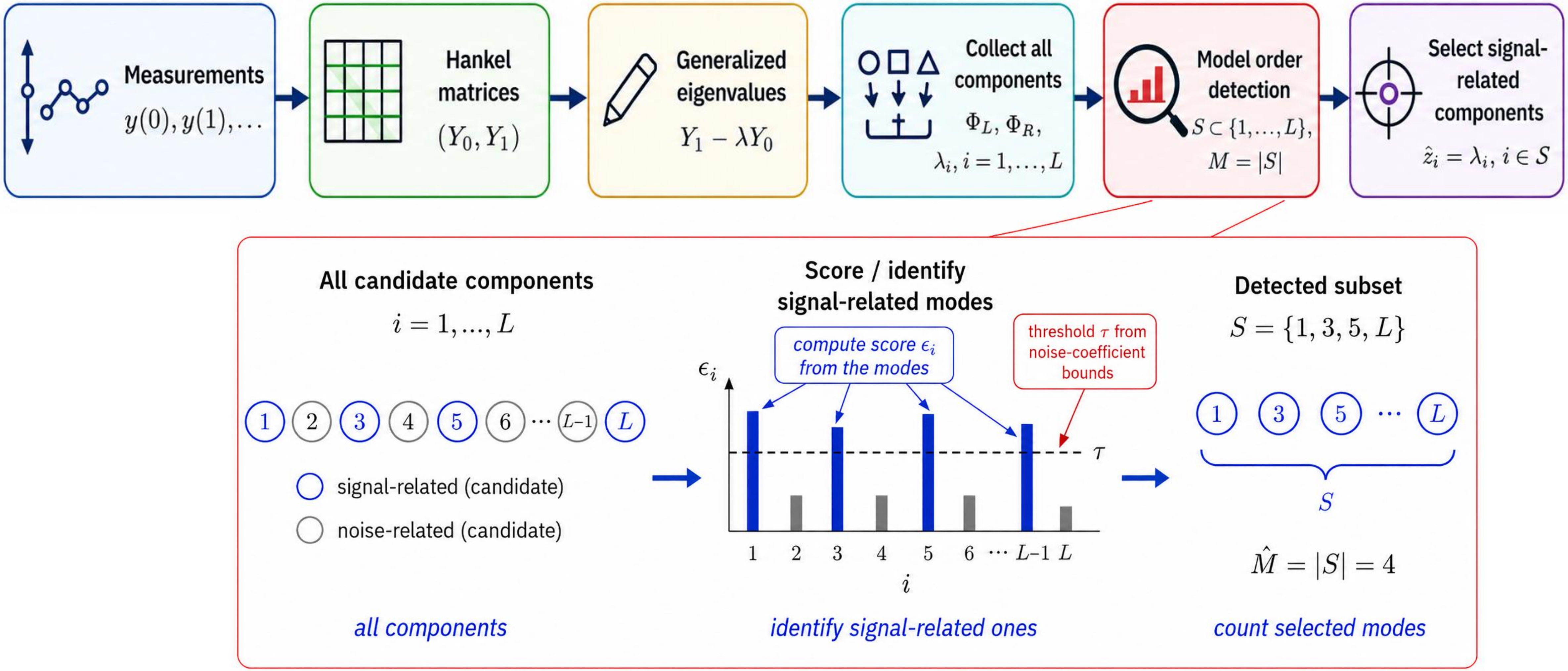
$$\Delta_i = \max_{m\neq i} \frac{1}{|z_i - z_m|}$$

$$\Phi_L = [\underbrace{\mathbf{Z}_L + \mathbf{E}_L}_{M\text{ columns}},\ \mathbf{C}_L]_{(N-L)\times L}$$

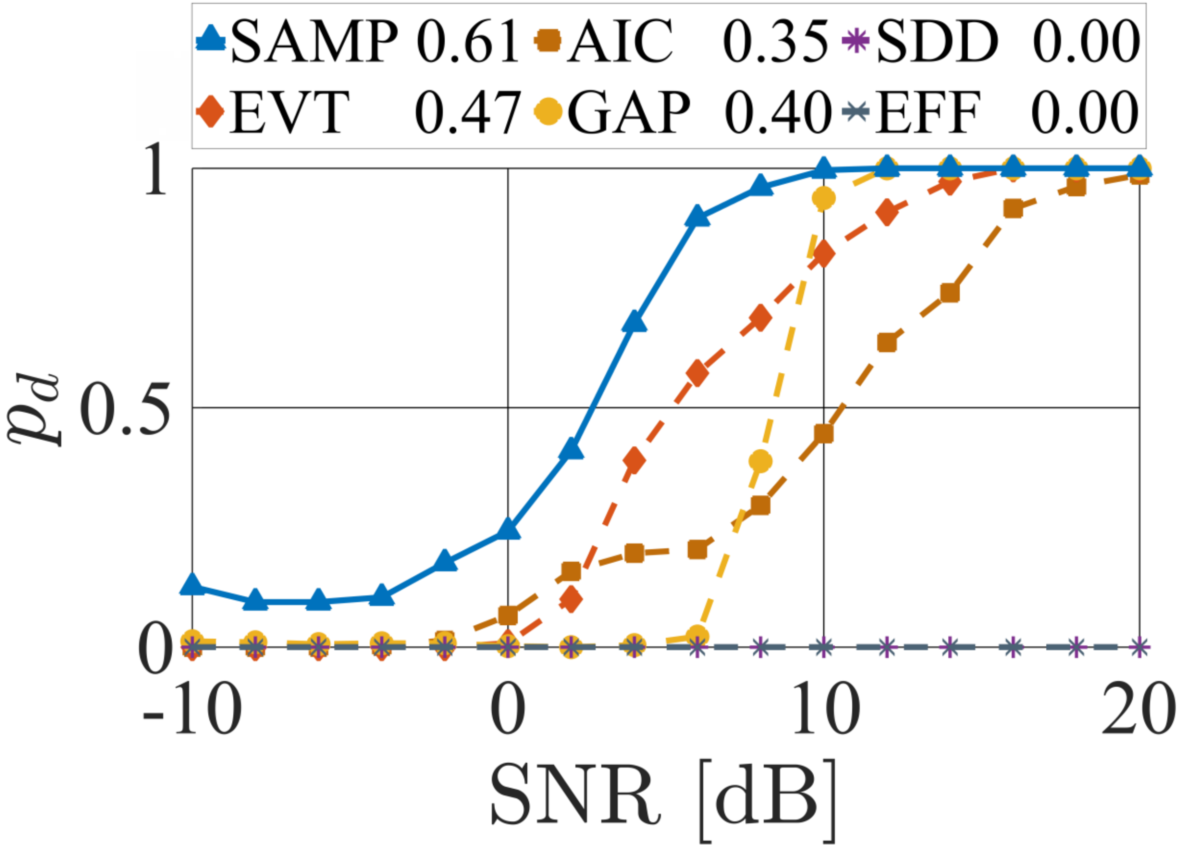
$$\left(\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ z_1^2 & z_2^2 & \cdots & z_M^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{N-L-1} & z_2^{N-L-1} & \cdots & z_M^{N-L-1} \end{bmatrix}}_{\mathbf{Z}_L} + \overbrace{\underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{E}_L^1 & \mathbf{E}_L^2 & \cdots & \mathbf{E}_L^M \\ | & | & & | \end{bmatrix}}_{\mathbf{E}_L}, \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{C}_L^1 & \mathbf{C}_L^2 & \cdots & \mathbf{C}_L^{L-M} \\ | & | & & | \end{bmatrix}}_{\mathbf{C}_L}}^{w(n)} \right)$$

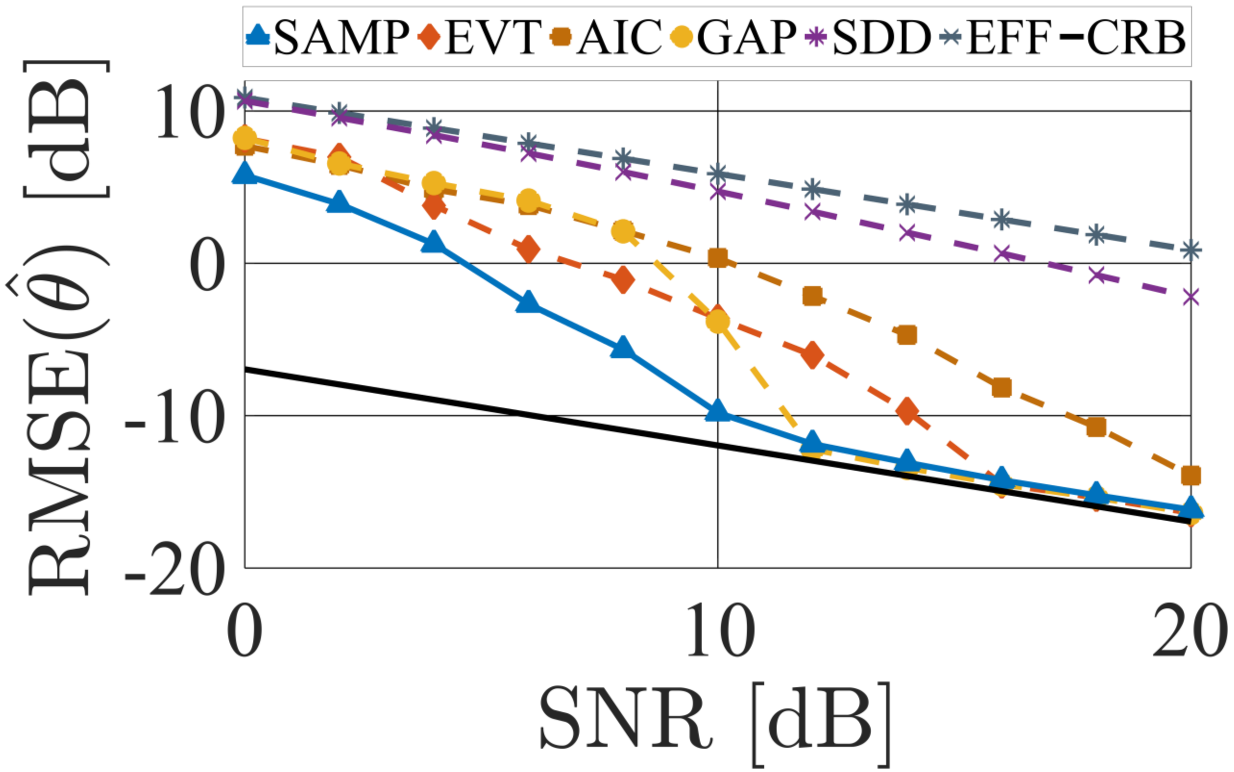
$$\underbrace{\mathbf{Z}_L + \mathbf{E}_L}_{\text{Signal-related modes}},\quad \underbrace{\mathbf{C}_L}_{\text{Noise-related spurious modes}}$$

Structure-Aware Matrix-Pencil Methode (SAMP)

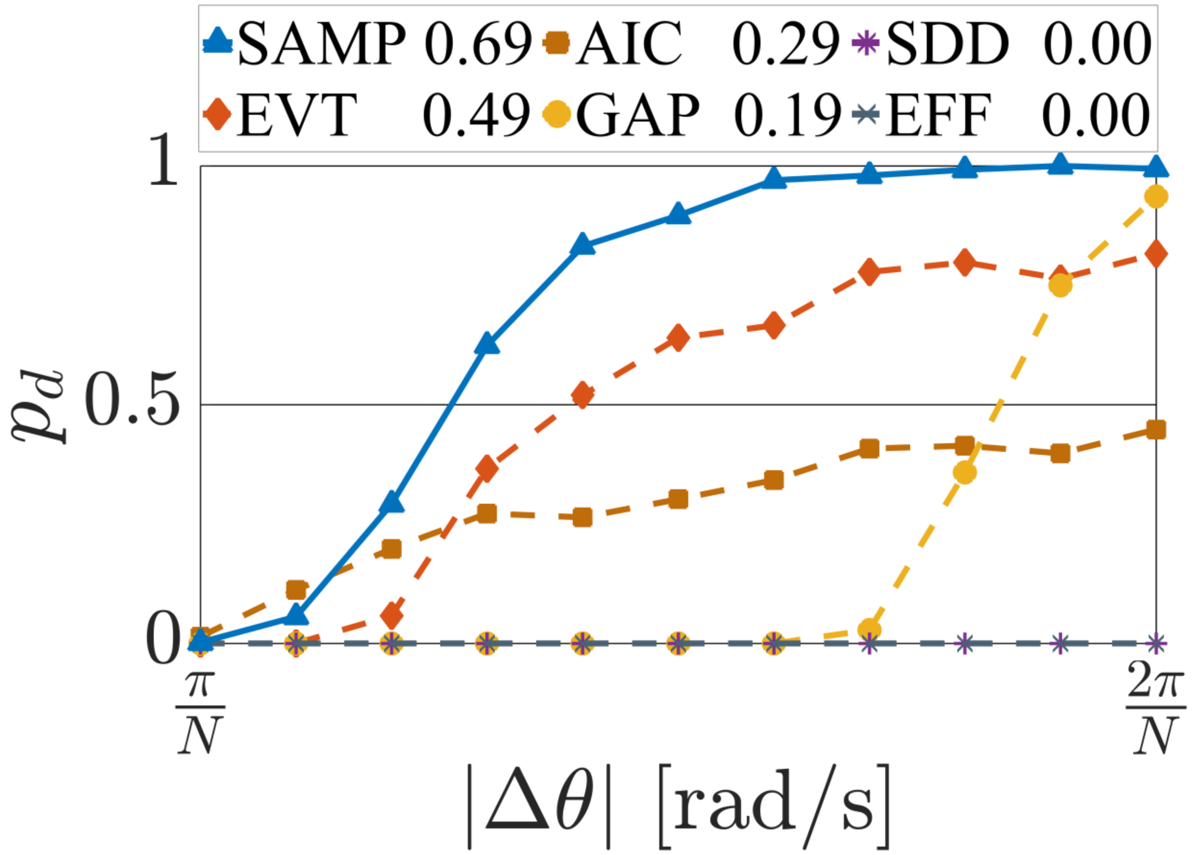


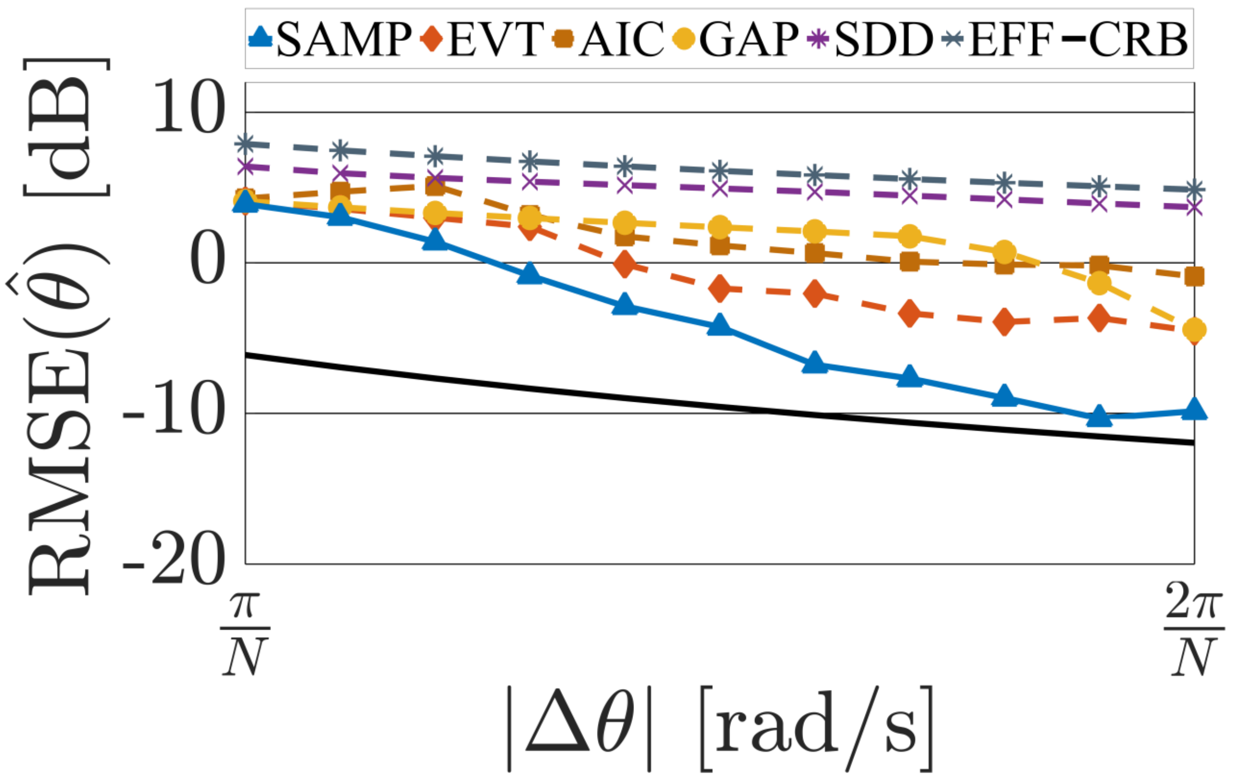
Simulation results





Simulation results



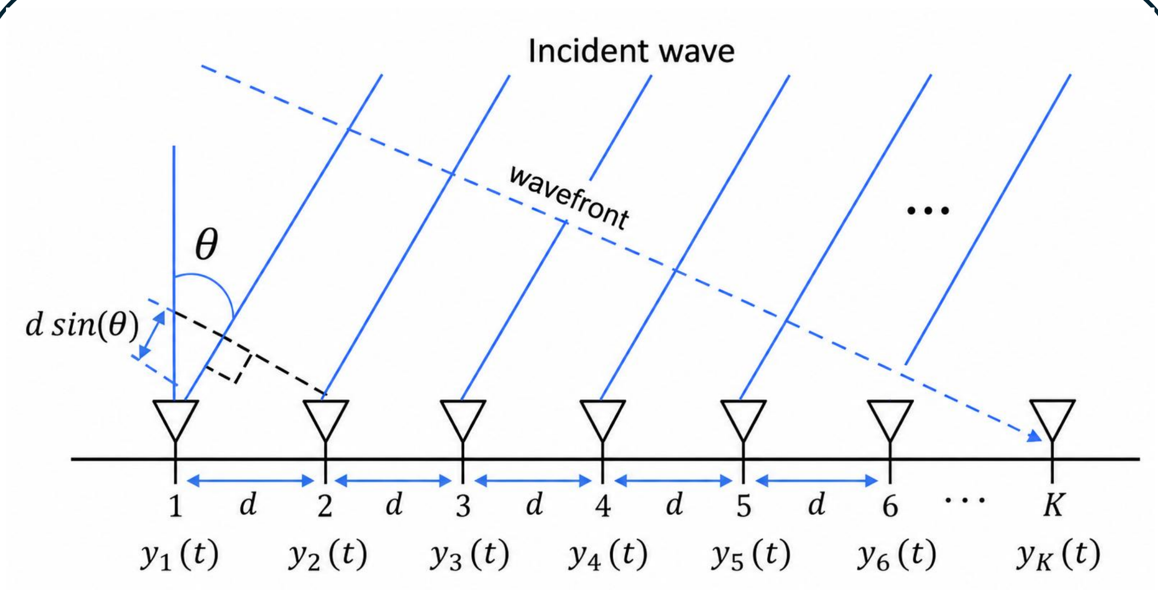


Spectral-Modes in Array Signal-Processing

We show that the same principle

estimate ⇒ select

can be applied to the task of DoA estimation using ESPRIT.

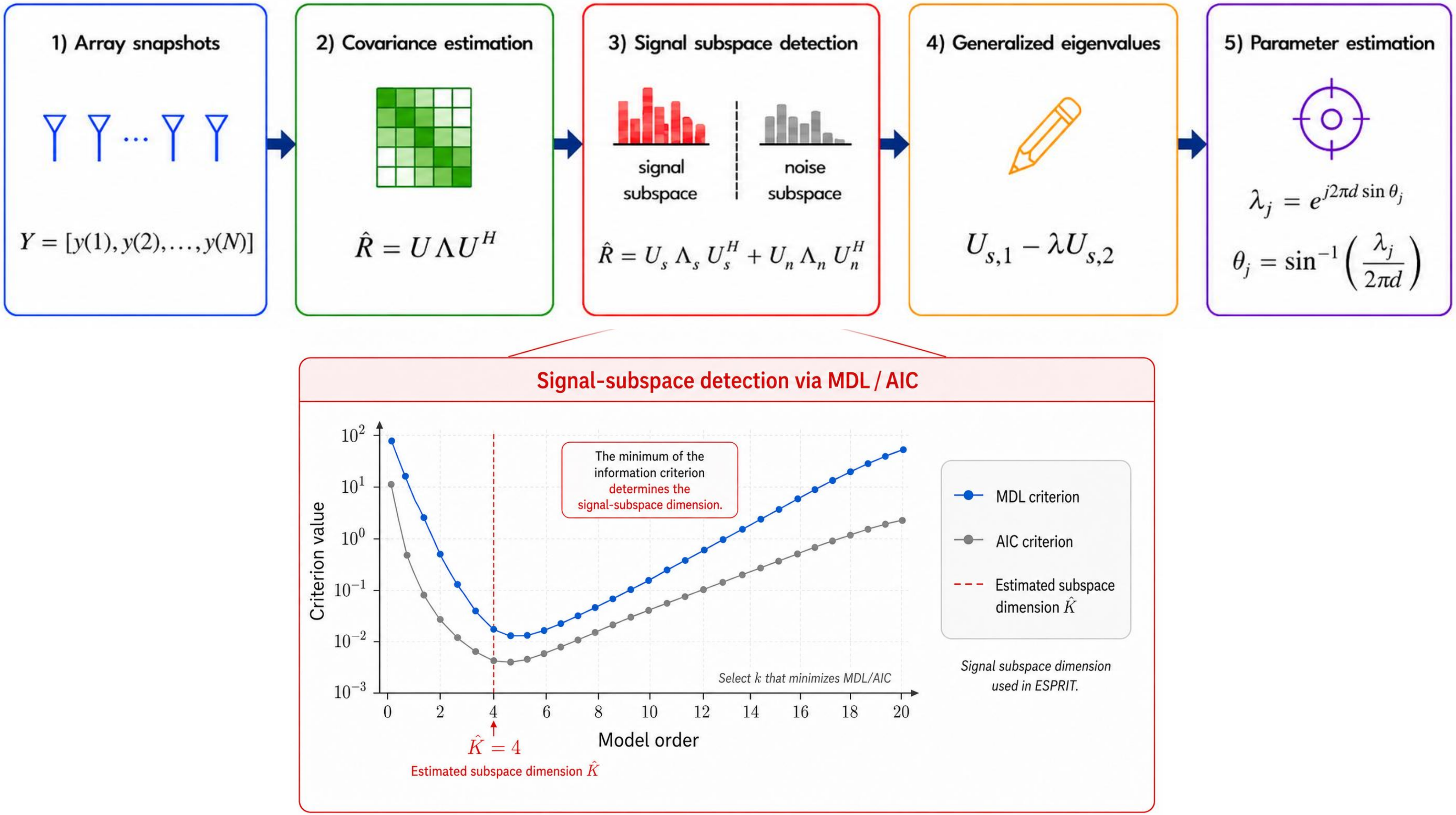


- The received signal at the k-th sensor is expressed by

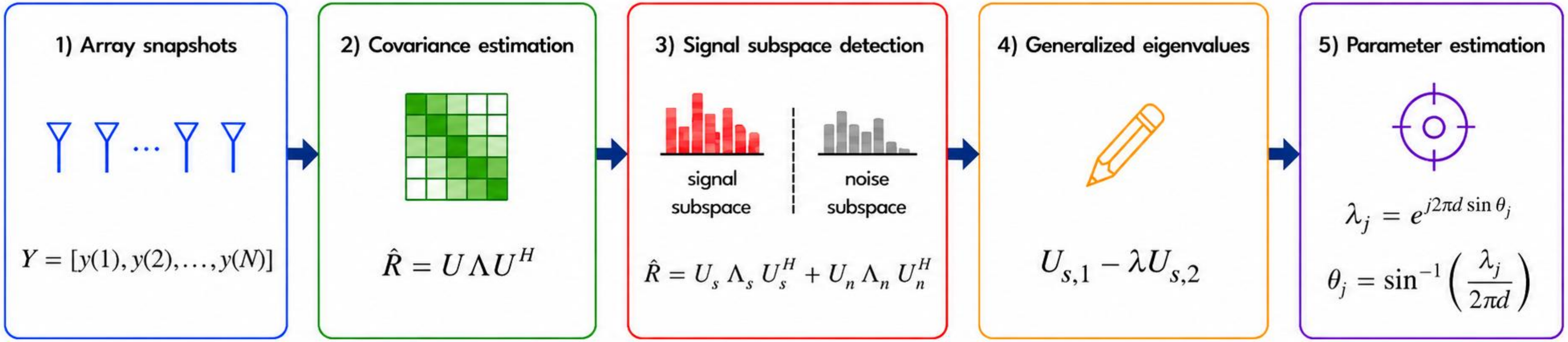
$$y_k(n) = \sum_{m=1}^{M} s_m(n) z_m^k + w_k(n),$$ where

$$z_m = e^{-j\frac{2\pi}{\lambda} d \sin\theta_m}$$

The ESPRIT approach



The ESPRIT approach



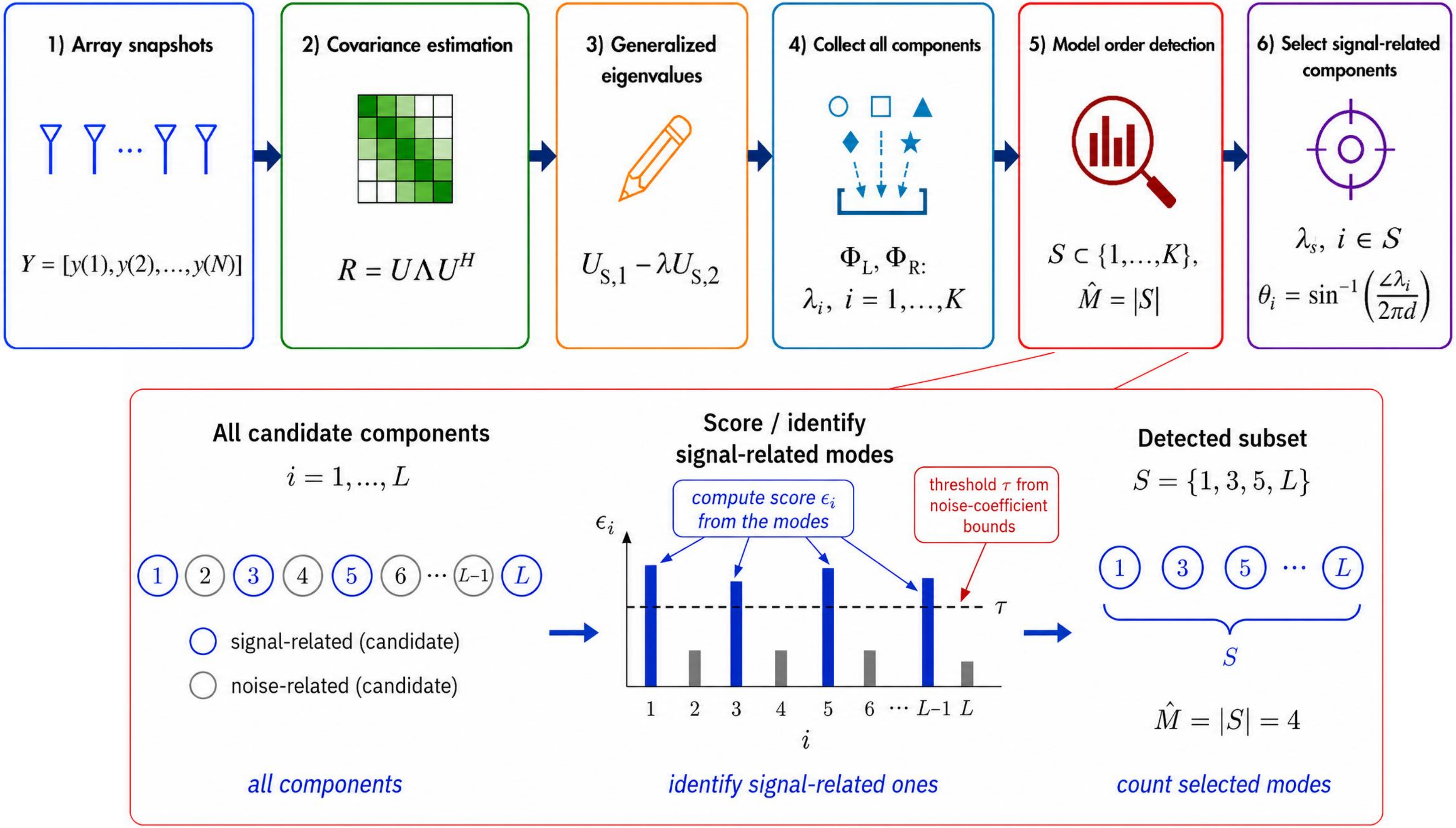
Generalized eigenvalues in the signal subspace

We show that:

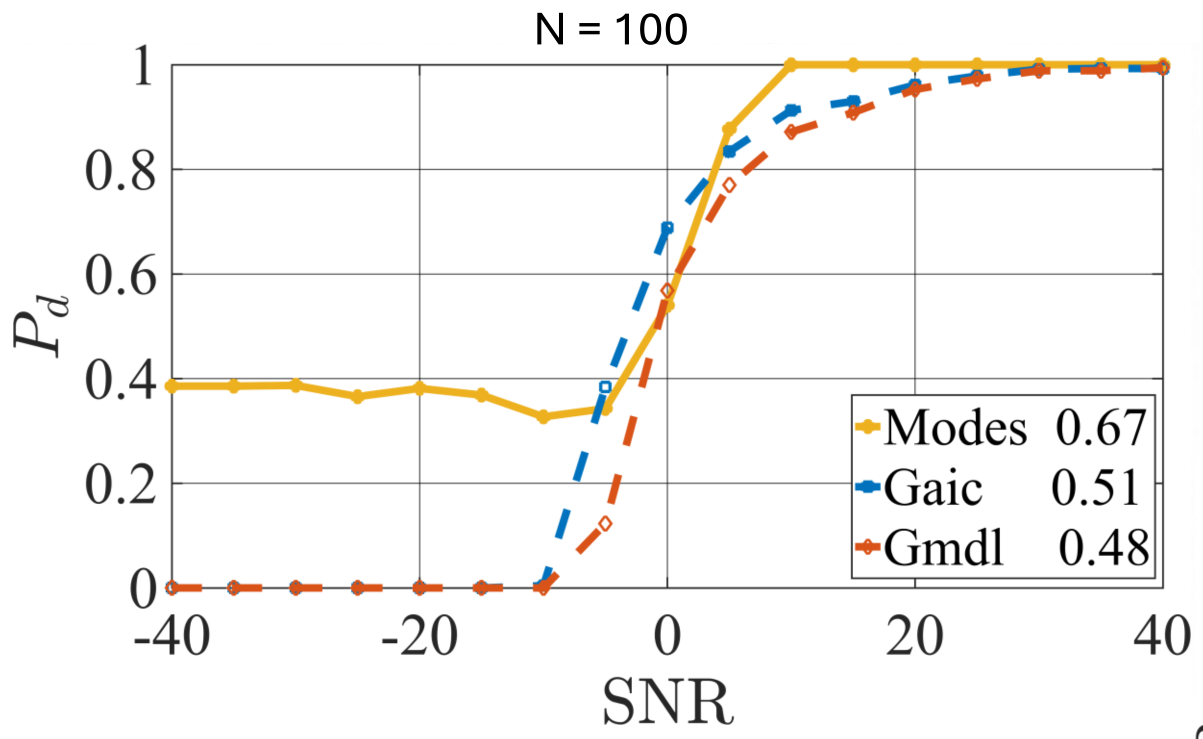
$$\boxed{\mathrm{U}_{s,1} - \lambda \mathrm{U}_{s,2} = (\mathrm{Y}_1 - \lambda \mathrm{Y}_0) C}$$

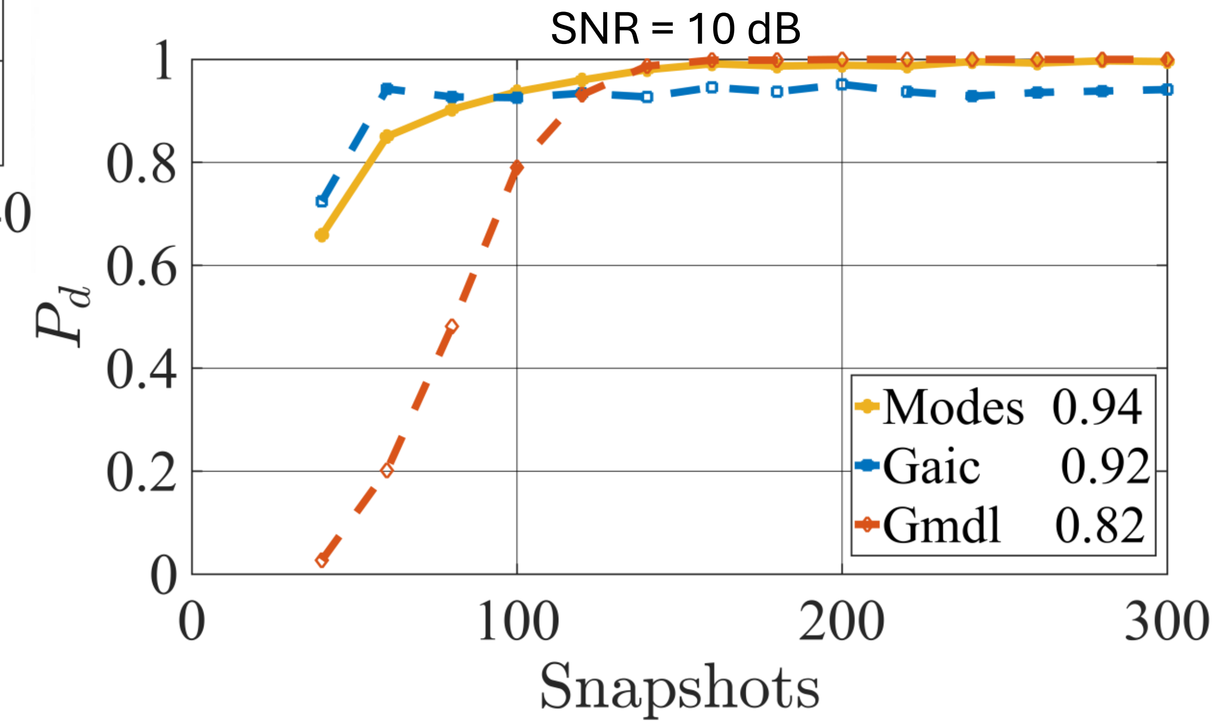
Thus, if C is full rank the signal-subspace eigenvalues and measurement-space eigenvalues coincide

Using the modes in ESPRIT

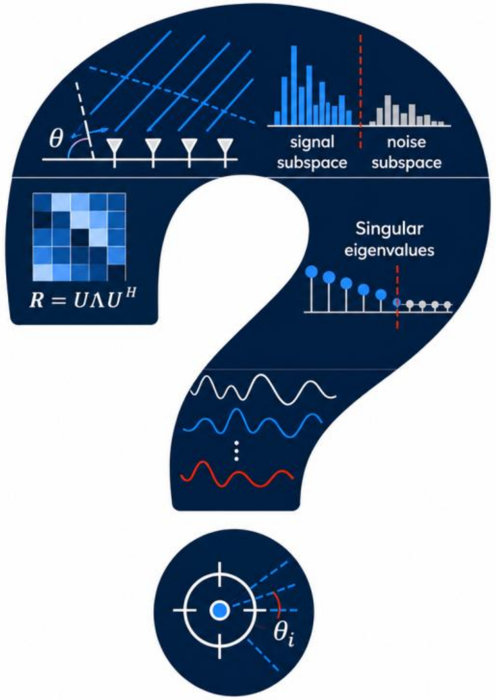


Preliminary results





Thank you !



θ
signal subspace
noise subspace
Singular eigenvalues
$R = U\Lambda U^H$
θ_i